Vector fields on surfaces

Sasha Patotski

Cornell University

ap744@cornell.edu

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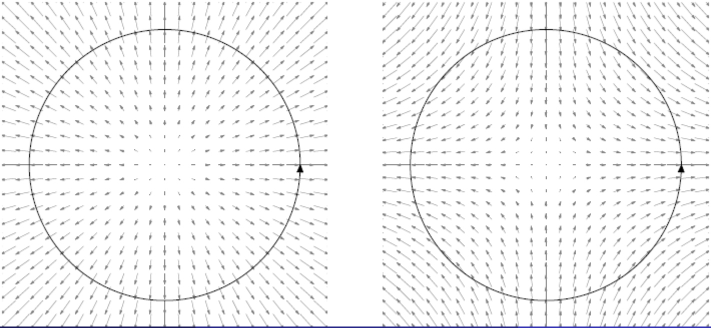
Some comments about the index

Definition

Index of an oriented curve is the total number of rotations the vectors do while going along the curve. We should count rotations with the "+" sign if they are in the same direction as the orientation of the curve, and with sign "-" otherwise.

Notice: then index **does not depend** on the orientation of the curve.



Index theorem

Theorem

Suppose we have a vector field on $\mathbb{R}^2$ and an oriented closed nonself intersecting curve C. Index of a curve C is equal to the sum of indices of singular points inside this curve.

Corollary

If index of a closed curve is **not** 0, *then there is a singular point inside.*

Applications

Theorem

Let $f : D \to \mathbb{R}^2$ be a continuous map from a disk to itself, such that each point of $S^1 = \partial D$ is mapped to itself. Then there exists a point $x \in D$ mapping to the center O of D.

Theorem (Fundamental theorem of algebra)

Any polynomial $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ with complex coefficients has a complex root.

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We identify $\mathbb{R}^2$ and $\mathbb{C}$. Consider two vector fields $v_z = z^n$ and $w_z = P(z)$.

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If $a = \max\{|a_0|, \ldots, |a_{n-1}|\}$, then (for $R > 1$)
$|w_z - v_z| = |a_{n-1}z^{n-1} + \cdots + a_0| \leq |a_{n-1}|R^{n-1} + \cdots + |a_n| \leq naR^{n-1}$.

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So vectors v_z and w_z can't point in opposite directions.

Thus index of the circle C with respect to v and w is **the same**.

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Index of the origin $0 \in \mathbb{C}$ *with respect to the vector field* $v_z = z^n$ *equals* n*.*

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Done!

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Vectors v_z and w_z can't point in opposite directions on a large enough circle. So indices of the circle $C = \{z \in \mathbb{C} \mid |z| = R\}$ w.r.t. v and w are the same.

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Lemma above gives that index of C w.r.t. v is $n \neq 0$.

So w has a singular point inside C, i.e. the polynomial $P(z)$ has a root inside C.

Vector fields

Definition

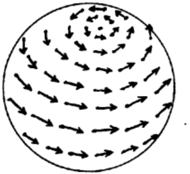
If Σ is a surface (in $\mathbb{R}^n$), then a vector field on Σ is a choice of a tangent vector at each point of the surface.

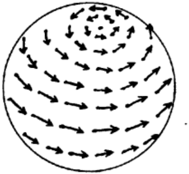
Definition

We can define singular points, indices of singular points the same way we defined these notions on the plane.

Note: not clear how to define index of a curve.

Example: Consider the standard sphere in $\mathbb{R}^3$ being rotated around the z-axis. Then velocities of points will give a vector field on the sphere.





Exercise: What are the singular points of the above vector field? what are their indices?
Exercise: Give an example of a vector field on a torus without singular points? can we find such a field on a sphere?

Theorem

Suppose we are given a vector field on a sphere with finitely many singular points. Then the sum of indices of all singular points equals 2*.*

Corollary

Any vector field on a sphere has a singular point.

Corollary

You can't comb a hairy ball. In particular, there always will be at least one hair perpendicular to the surface.

Proof:

Suppose you can. At each point of the surface of the ball, take the tangent plane at this point.
Project the hair at that point to the plane. Since it wasn't perpendicular, the projection is not 0.
So we've got a vector field with no singular points.
Contradiction!

Proof of the theorem

Theorem

Suppose we are given a vector field on a sphere with finitely many singular points. Then the sum of indices of all singular points equals 2.

Proof: